ANTI-POWER j-FIXES OF THE THUE-MORSE WORD

Marisa Gaetz

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Abstract. Recently, Fici, Restivo, Silva, and Zamboni introduced the notion of a k-anti-power, which is defined as a word of the form $w^{(1)}w^{(2)}\cdots w^{(k)}$, where $w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ are distinct words of the same length. For an infinite word w and a positive integer k, define $AP_j(w,k)$ to be the set of all integers m such that $w_{j+1}w_{j+2}\cdots w_{j+km}$ is a k-anti-power, where w_i denotes the i-th letter of w. Define also $\mathcal{F}_j(k) = (2\mathbb{Z}^+ - 1) \cap AP_j(\mathbf{t}, k)$, where $\mathbf{t}$ denotes the Thue-Morse word. For all $k \in \mathbb{Z}^+$, $\gamma_j(k) = \min(AP_j(\mathbf{t}, k))$ is a well-defined positive integer, and for $k \in \mathbb{Z}^+$ sufficiently large, $\Gamma_j(k) = \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(k))$ is a well-defined odd positive integer. In his 2018 paper, Defant shows that $\gamma_0(k)$ and $\Gamma_0(k)$ grow linearly in k. We generalize Defant's methods to prove that $\gamma_j(k)$ and $\Gamma_j(k)$ grow linearly in k for any nonnegative integer j. In particular, we show that $1/10 \leq \liminf_{k\to\infty}(\gamma_j(k)/k) \leq 9/10$ and $1/5 \leq \limsup_{k\to\infty}(\gamma_j(k)/k) \leq 3/2$. Additionally, we show that $\liminf_{k\to\infty}(\Gamma_j(k)/k) = 3/2$ and $\limsup_{k\to\infty}(\Gamma_j(k)/k) = 3$.

1. Introduction

A finite word is called a *k-power* if it is of the form w^k for some word w. A particularly famous consequence of the study of k-powers is Axel Thue's 1912 paper [14], which introduces an infinite binary word that does not contain any 3-powers as subwords. This word has since caught the interest of numerous academicians [1, 2, 4, 6–9, 11–13] spanning the fields of combinatorics, analytic number theory [1], game theory [7], and economics [13]. It is now known as the Thue-Morse word.

Definition 1.1. Let $A_0 = 0$. For each nonnegative integer n, let $B_n = \overline{A_n}$ be the Boolean complement of A_n, and let $A_{n+1} = A_nB_n$. The *Thue-Morse word* $\mathbf{t}$ is defined as

$$\mathbf{t} = \lim_{n\to\infty} A_n = 0110100110010110\cdots.$$

E-mail address: mgaetz@mit.edu.

As a natural adaptation of the Ramsey-type notion of a k-power, Fici, Restivo, Silva, and Zamboni [10] introduce the anti-Ramsey-type notion of a k-anti-power. A *k-anti-power* is a word w of the form $w = w^{(1)}w^{(2)} \cdots w^{(k)}$, where $w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ are distinct words of the same length. For example, 110100 is a 3-anti-power, while 101011 is not. Since the introduction of this notion in 2016, k-anti-powers have received much attention [3, 5, 8, 12].

As their main result, Fici et al. show that every infinite word contains powers of any order or anti-powers of any order. In doing so, they define the following set, which corresponds to an infinite word w and a positive integer k:

$$AP(w, k) = \{m \in \mathbb{Z}^+ \mid w_1 w_2 \cdots w_{km} \text{ is a } k\text{-anti-power}\}.$$

Here, w_i indicates the i-th letter of the infinite word w. Such subwords (i.e. those starting from the first index of w) are called *prefixes* of w. In [8], Defant introduces the generalized definition

$$AP_j(w, k) = \{m \in \mathbb{Z}^+ \mid w_{j+1} w_{j+2} \cdots w_{j+km} \text{ is a } k\text{-anti-power}\},$$

himself studying $AP_0(\mathbf{t}, k) = AP(\mathbf{t}, k)$. Subwords beginning at the $(j+1)$-st index of a word w will be referred to as *j-fixes* of w. An easy consequence of [10, Theorem 6] is that $AP_j(\mathbf{t}, k)$ is nonempty for any nonnegative integer j and all positive integers k. We can therefore make the following definition:

Definition 1.2. Let $\gamma_j(k) = \min(AP_j(\mathbf{t}, k))$.

For $j = 0$, it is the case that $m \in AP_0(\mathbf{t}, k)$ if and only if $2m \in AP_0(\mathbf{t}, k)$ (see Remark 2.1). As a consequence, the only interesting elements of $AP_0(\mathbf{t}, k)$ are those that are odd. Thus, Defant [8] makes the following definition for $j = 0$ (which we have written in terms of arbitrary $j \in \mathbb{Z}^{\geq 0}$):

Definition 1.3. Let $\mathcal{F}_j(k)$ denote the set of odd positive integers m such that the j-fix of $\mathbf{t}$ of length km is a k-anti-power. Let $\Gamma_j(k) = \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(k))$.

For sufficiently large k, $\Gamma_j(k)$ is a well-defined odd positive integer (see Remark 4.6). However, if $j \neq 0$, it is not necessarily the case that $m \in AP_j(\mathbf{t}, k)$ if and only if $2m \in AP_j(\mathbf{t}, k)$. For example, $4 \in AP_2(\mathbf{t}, 3)$, whereas $2 \notin AP_2(\mathbf{t}, 3)$. As such, in Section 4, we will discuss our motivation for defining $\Gamma_j(\mathbf{t}, k)$ in this way.

Remark 1.4. It is immediate from Definition 1.3 that $\mathcal{F}_j(1) \supseteq \mathcal{F}_j(2) \supseteq \mathcal{F}_j(3) \supseteq \cdots$ for any $j \in \mathbb{Z}^{\geq 0}$. It follows that $\gamma_j(1) \leq \gamma_j(2) \leq \gamma_j(3) \leq \cdots$ and that $\Gamma_j(k)$ is nondecreasing when it is finite.

As a means to understanding $\gamma_j(k)$ and $\Gamma_j(k)$, it will often be useful to consider the following related function:

Definition 1.5. For a positive integer m, let $\mathfrak{K}_j(m)$ denote the smallest positive integer k such that the j-fix of $\mathbf{t}$ of length km is not a k-anti-power.

A simple application of the Pigeonhole Principle gives that $\mathfrak{K}_j(m) \leq 2^m + 1$. However, Defant [8] and Narayanan [12] prove significantly better bounds on $\mathfrak{K}_0(m)$, showing it grows linearly in m. Using these bounds, Defant [8] is ultimately able to show the following:

Theorem 1.6 ([8])**.**

- $\frac{1}{4}^* \leq \liminf_{k\to\infty} \frac{\gamma_0(k)}{k} \leq \frac{9}{10}$
- $\frac{1}{2}^\dagger \leq \limsup_{k\to\infty} \frac{\gamma_0(k)}{k} \leq \frac{3}{2}$
- $\liminf_{k\to\infty} \frac{\Gamma_0(k)}{k} = \frac{3}{2}$
- $\limsup_{k\to\infty} \frac{\Gamma_0(k)}{k} = 3.$

Narayanan [12] improves the above asymptotic bounds in the following way:

Theorem 1.7 ([12])**.**

- $\frac{3}{4} \leq \liminf_{k\to\infty} \frac{\gamma_0(k)}{k} \leq \frac{9}{10}$
- $\limsup_{k\to\infty} \frac{\gamma_0(k)}{k} = \frac{3}{2}.$

The goal of this paper is to demonstrate similarly good bounds on the asymptotic growth of $\gamma_j(k)$ and $\Gamma_j(k)$ for general j. To do so, we will roughly follow the outline of Defant's paper [8], generalizing his bounds for $\mathfrak{K}_0(m)$ to bounds for $\mathfrak{K}_j(m)$; this will in turn allow us to prove that $\gamma_j(k)$ and $\Gamma_j(k)$ grow linearly in k. Specifically, we aim to prove the following:

- $\frac{1}{10} \leq \liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{9}{10}$
- $\frac{1}{5} \leq \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{3}{2}$
- $\liminf_{k\to\infty} \frac{\Gamma_j(k)}{k} = \frac{3}{2}$
- $\limsup_{k\to\infty} \frac{\Gamma_j(k)}{k} = 3.$

*Erroneously stated in [8] as 1/2 (as will later be explained)

†Erroneously stated in [8] as 1 (as will later be explained)

Remark 1.8. Note that we follow the methods of Defant [8] rather than those of Narayanan [12], which seem more difficult to generalize to arbitrary $j \in \mathbb{Z}^{\geq 0}$.

In Section 2, we cover preliminary results relating to the Thue-Morse word. In Section 3 (resp. Section 4), we prove the aforementioned asymptotic bounds on $\gamma_j(k)/k$ (resp. $\Gamma_j(k)/k$).

2. Properties of the Thue-Morse Word

In this section, we will discuss some properties of the Thue-Morse word $\mathbf{t} = \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\cdots$ that will be of use throughout the remainder of the paper. It is well known that the i-th letter $\mathbf{t}_i$ of the Thue-Morse word has the same parity as the number of 1's in the binary expansion of $i-1$. In his 1912 paper [14], Thue proved that $\mathbf{t}$ is *overlap-free*, meaning that if x and y are finite words (with x nonempty), then $\mathbf{t}$ does not contain $xyxyx$ as a subword. Taking y to be empty shows that $\mathbf{t}$ does not contain any 3-powers as subwords.

Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be sets of words. We say a function $f : \mathcal{W}_1 \to \mathcal{W}_2$ is a *morphism* if $f(xy) = f(x)f(y)$ for all words $x, y \in \mathcal{W}_1$. We will write $\mathbb{A}^{\leq\omega}$ to refer to the set of all words over an alphabet $\mathbb{A}$. Using this notation, let $\mu : \{0,1\}^{\leq\omega} \to \{01, 10\}^{\leq\omega}$ be the morphism uniquely defined by $\mu(0) = 01$ and $\mu(1) = 10$. Similarly, let $\sigma : \{01, 10\}^{\leq\omega} \to \{0,1\}^{\leq\omega}$ be the morphism uniquely defined by $\sigma(01) = 0$ and $\sigma(10) = 1$. The Thue-Morse word $\mathbf{t}$ and its Boolean complement $\overline{\mathbf{t}}$ are the unique one-sided infinite words over the alphabet $\{0,1\}$ that are fixed by μ. Similarly, $\mathbf{t}$ and $\overline{\mathbf{t}}$, as viewed over the alphabet $\{01, 10\}$, are the unique one-sided infinite words fixed by σ. The observation that $\mu(\mathbf{t}) = \mathbf{t}$ allows us to view $\mathbf{t}$ as a word over the alphabet $\{01, 10\}$. More generally, if we recall the definitions of A_n and B_n from Definition 1.1 and note the equalities $A_n = \mu^n(0)$ and $B_n = \mu^n(1)$, we can view $\mathbf{t}$ as a word over the alphabet $\{A_n, B_n\}$.

Remark 2.1. Using that $\mu(\mathbf{t}) = \mathbf{t}$ and $\sigma(\mathbf{t}) = \mathbf{t}$, it is straightforward to see that $m \in AP_0(\mathbf{t}, k)$ if and only if $2m \in AP_0(\mathbf{t}, k)$.

We will follow Defant [8] in using the notation $\langle \alpha, \beta \rangle = \mathbf{t}_\alpha \mathbf{t}_{\alpha+1} \cdots \mathbf{t}_\beta$ for any positive integers α, β with $\alpha \leq \beta$. We are now in a position to establish some preliminary results relating to $\mathbf{t}$.

Fact 2.2 ([8])**.** *For any positive integers n and r, $\langle 2^n r + 1, 2^n(r+1) \rangle = \mu^n(\mathbf{t}_{r+1})$.*

Lemma 2.3. *For $m \in \mathbb{Z}^+$, $\mathbf{t}_{2m+1} \neq \mathbf{t}_{2m+2}$.*

Proof. If $\mathbf{t}_{m+1} = 1$, then $\mu(\mathbf{t}_{m+1}) = \mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$. Similarly, if $\mathbf{t}_{m+1} = 0$, then $\mu(\mathbf{t}_{m+1}) = \mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 01$. In either case, $\mathbf{t}_{2m+1} \neq \mathbf{t}_{2m+2}$. □

Lemma 2.4. *Let $L, k \in \mathbb{Z}^+$. Then $\mathbf{t}_{2^L k+1}\mathbf{t}_{2^L k+2} = \mathbf{t}_{2^{L+1}k+1}\mathbf{t}_{2^{L+1}k+2}$.*

Proof. We proceed by induction on L. Fix some $k \in \mathbb{Z}^+$ and consider the case where $L = 1$. We seek to show that $\mathbf{t}_{2k+1}\mathbf{t}_{2k+2} = \mathbf{t}_{4k+1}\mathbf{t}_{4k+2}$. Suppose that $\mathbf{t}_{k+1} = 1$; the case in which $\mathbf{t}_{k+1} = 0$ can be done similarly. Note that $\mu(\mathbf{t}_{k+1}) = \mathbf{t}_{2k+1}\mathbf{t}_{2k+2} = 10$. Similarly, $\mu(\mathbf{t}_{2k+1}) = \mathbf{t}_{4k+1}\mathbf{t}_{4k+2} = 10$. So we have that $\mathbf{t}_{2k+1}\mathbf{t}_{2k+2} = 10 = \mathbf{t}_{4k+1}\mathbf{t}_{4k+2}$, as desired.

Now, suppose that $\mathbf{t}_{2^{L-1}k+1}\mathbf{t}_{2^{L-1}k+2} = \mathbf{t}_{2^Lk+1}\mathbf{t}_{2^Lk+2}$ for some arbitrary $L \in \mathbb{Z}^+$. Then $\mu(\mathbf{t}_{2^{L-1}k+1}) = \mathbf{t}_{2^Lk+1}\mathbf{t}_{2^Lk+2} = \mu(\mathbf{t}_{2^Lk+1}) = \mathbf{t}_{2^{L+1}k+1}\mathbf{t}_{2^{L+1}k+2}$. Therefore, $\mathbf{t}_{2^Lk+1}\mathbf{t}_{2^Lk+2} = \mathbf{t}_{2^{L+1}k+1}\mathbf{t}_{2^{L+1}k+2}$. The lemma follows by induction. □

3. Asymptotics for $\gamma_j(k)$

In this section, we prove that $\frac{1}{10} \le \liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \le \frac{9}{10}$ and $\frac{1}{5} \le \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \le \frac{3}{2}$.

3.1. **Lower Bounds for $\gamma_j(k)/k$.** In this subsection, we present a series of lemmas that collectively establish an upper bound for $\mathfrak{K}_j(m)$ for any integer $m \ge 2$. This will allow us to establish lower bounds for $\liminf_{k\to\infty}(\gamma_j(k)/k)$ and $\limsup_{k\to\infty}(\gamma_j(k)/k)$. We begin with three lemmas that we will apply in the proofs of many of the lemmas later in this subsection.

Lemma 3.1. *Let $m, j \in \mathbb{Z}^{\ge 0}$ with $m \ge 2$, and let $\ell = \lceil \log_2(m+j) \rceil$. For any $s, a \in \mathbb{Z}^+$, there exists a nonnegative integer r such that*

$$\langle 2^\ell(s-1)+1, 2^\ell(s+a)\rangle = w\langle rm+j+1, (r+1)m+j\rangle z$$

for some words w and z (with z nonempty).

Proof. Fix some $s, a \in \mathbb{Z}^+$. Note that

$$|\langle 2^\ell(s-1)+1, 2^\ell(s+a)\rangle| = 2^\ell(a+1) \ge 2^{\ell+1} \ge 2(m+j) \ge 2m.$$

Since $|\langle rm+j+1, (r+1)m+j\rangle| = m$ for any integer r, it follows that there exists $r \in \mathbb{Z}$ satisfying

$$2^\ell(s-1)+1 \le rm+j+1 < (r+1)m+j < 2^\ell(s+a). \tag{1}$$

Moreover, we can always choose r to be nonnegative; to verify this fact, it suffices to check that $r = 0$ satisfies (1) when $s = 1$:

$$2^\ell(s-1)+1 = 1 \le j+1 < m+j < 2^{\ell+1} \le 2^\ell(s+a).$$

When $s \ge 2$, any integer r satisfying (1) is clearly positive. □

Lemma 3.2. *Let $j \in \mathbb{Z}^{\ge 0}$, $m \in \mathbb{Z}^+$, and $\ell = \lceil \log_2(m+j) \rceil$. If $\mathfrak{K}_j(m) > 2^\ell + 1$, then $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$ and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$.*

Proof. Suppose $\mathfrak{K}_j(m) > 2^\ell + 1$. Let $w_0 = \langle j+1, m+j\rangle$, $w_1 = \langle 2^{\ell-1}m + j + 1, (2^{\ell-1}+1)m + j\rangle$, and $w_2 = \langle 2^\ell m + j + 1, (2^\ell + 1)m + j\rangle$. By our assumption that $\mathfrak{K}_j(m) > 2^\ell + 1$, we have that w_0, w_1, and w_2 are distinct. Notice that for each $n \in \{0, 1, 2\}$, the word w_n is a j-fix of

$$\langle nm2^{\ell-1} + 1, (nm+2)2^{\ell-1}\rangle = \mu^{\ell-1}(\mathbf{t}_{nm+1}\mathbf{t}_{nm+2}).$$

It follows that $\mathbf{t}_1\mathbf{t}_2$, $\mathbf{t}_{m+1}\mathbf{t}_{m+2}$, and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2}$ are distinct. Note that $\mathbf{t}_1\mathbf{t}_2 = 01$ and that $\mathbf{t}_{2m+1} \neq \mathbf{t}_{2m+2}$ (by Lemma 2.3); hence, $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$. Therefore, $\mu(\mathbf{t}_{m+1}) = \mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$, which implies that $\mathbf{t}_{m+1} = 1$. Consequently, $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$. □

Lemma 3.3. *Let $j, m \in \mathbb{Z}^{\geq 0}$ with $m \geq 2$, and let $\ell = \lceil \log_2(m+j)\rceil$. Suppose there exists $s \in \mathbb{Z}^+$ such that $\mathbf{t}_s\mathbf{t}_{s+1} = \mathbf{t}_{m+s}\mathbf{t}_{m+s+1}$. Then*

$$\mathfrak{K}_j(m) < 2^\ell + \frac{2^\ell(s+1) - j}{m}.$$

Proof. Observe that

$$\langle 2^\ell(s-1)+1, 2^\ell(s+1)\rangle = \mu^\ell(\mathbf{t}_s\mathbf{t}_{s+1}) = \mu^\ell(\mathbf{t}_{m+s}\mathbf{t}_{m+s+1}) = \langle 2^\ell(m+s-1)+1, 2^\ell(m+s+1)\rangle.$$

Applying Lemma 3.1 with $a = 1$ gives that there exists $r \in \mathbb{Z}^{\geq 0}$ such that

$$\langle 2^\ell(s-1)+1, 2^\ell(s+1)\rangle = w\langle rm+j+1, (r+1)m+j\rangle z \tag{2}$$

for some words w and z (with z nonempty). Adding $2^\ell m$ to each index in (2) shows that there exist words w' and z' (with z' nonempty) for which

$$\langle 2^\ell(m+s-1)+1, 2^\ell(m+s+1)\rangle = w'\langle (2^\ell+r)m+j+1, (2^\ell+r+1)m+j\rangle z'. \tag{3}$$

Notice that $|w'| = rm + j - 2^\ell(s-1) = |w|$. Equations (2) and (3) therefore imply

$$\langle rm+j+1, (r+1)m+j\rangle = \langle (2^\ell+r)m+j+1, (2^\ell+r+1)m+j\rangle.$$

Using (2) to see that $r + 1 < \dfrac{2^\ell(s+1)-j}{m}$, we therefore have that

$$\mathfrak{K}_j(m) \leq 2^\ell + r + 1 < 2^\ell + \frac{2^\ell(s+1)-j}{m},$$

as desired. □

Now that we have established the preceding preliminary results, we are ready to derive upper bounds for $\mathfrak{K}_j(m)$ for all integers $m \geq 2$. We consider the cases $m \equiv 0 \pmod 2$, $m \equiv 1 \pmod 8$, $m \equiv 29 \pmod{32}$, and remaining values of m. We then combine the bounds derived in each of these cases into a uniform upper bound on $\mathfrak{K}_j(m)$. We first consider the case in which $m \equiv 0 \pmod 2$.

Lemma 3.4. *Let $m = 2^L k$, where $L, k \in \mathbb{Z}^+$. Let $j \in \mathbb{Z}^{\geq 0}$, and let $\ell = \lceil \log_2(m+j) \rceil$. Then*

$$\mathfrak{K}_j(m) < 2^{\ell+1} + \frac{2^{\ell+1} - j}{m}.$$

Proof. By Lemma 2.4, we have that $\mathbf{t}_{2^L k+1}\mathbf{t}_{2^L k+2} = \mathbf{t}_{2^{L+1}k+1}\mathbf{t}_{2^{L+1}k+2}$. It follows that $\langle 2^\ell m+1, 2^\ell(m+2)\rangle = \mu^\ell(\mathbf{t}_{m+1}\mathbf{t}_{m+2}) = \mu^\ell(\mathbf{t}_{2m+1}\mathbf{t}_{2m+2}) = \langle 2^{\ell+1}m+1, 2^{\ell+1}(m+1)\rangle$. Applying Lemma 3.1 with $s = 1$ and $a = 1$ shows that there exists $r \in \mathbb{Z}^{\geq 0}$ such that

$$\langle 1, 2^{\ell+1}\rangle = w\langle rm+j+1, (r+1)m+j\rangle z \tag{4}$$

for some words w and z (with z nonempty). Adding $2^\ell m$ to each index in (4) gives that

$$\langle 2^\ell m+1, 2^\ell(m+2)\rangle = w'\langle (2^\ell+r)m+j+1, (2^\ell+r+1)m+j\rangle z' \tag{5}$$

for some words w' and z' (with z' nonempty). Similarly, adding $2^{\ell+1}m$ to each index in (4) gives that

$$\langle 2^{\ell+1}m+1, 2^{\ell+1}(m+1)\rangle = w''\langle (2^{\ell+1}+r)m+j, (2^{\ell+1}+r+1)m+j\rangle z'' \tag{6}$$

for some words w'' and z'' (with z'' nonempty). Observe that $|w''| = rm+j = |w'|$. Equations (5) and (6) therefore give that

$$\langle (2^\ell+r)m+j+1, (2^\ell+r+1)m+j\rangle = \langle (2^{\ell+1}+r)m+j+1, (2^{\ell+1}+r+1)m+j\rangle.$$

Using (4) to note that $r+1 < \dfrac{2^{\ell+1}-j}{m}$, we get

$$\mathfrak{K}_j(m) \leq 2^{\ell+1} + r + 1 < 2^{\ell+1} + \frac{2^{\ell+1}-j}{m},$$

as desired. □

The following two lemmas establish upper bounds for $\mathfrak{K}_j(m)$ when $m \equiv 1 \pmod 8$. Setting $j = 0$ in Lemma 3.5 implies Defant's result [8, Lemma 15], while setting $j = 0$ in Lemma 3.7 gives a bound for $\mathfrak{K}_0(m)$ that is worse than the one given in [8, Lemma 16] by a factor of two.

Lemma 3.5. *Let $j \in \mathbb{Z}^{\geq 0}$, and suppose $m = 2^L h + 1$, where L and h are integers with $L \geq 3$ and h odd. Let $\ell = \lceil \log_2(m+j) \rceil$. We have*

$$\mathfrak{K}_j(m) < 2^\ell + \frac{2^\ell(2^{L+1}+4) - j}{m}.$$

Proof. Suppose instead that $\mathfrak{K}_j(m) \geq 2^\ell + \dfrac{2^\ell(2^{L+1}+4)-j}{m}$. We will obtain a contradiction to Lemma 3.3 by finding a positive integer $s \leq 2^{L+1}+3$ satisfying $\mathbf{t}_s\mathbf{t}_{s+1} = \mathbf{t}_{m+s}\mathbf{t}_{m+s+1}$. Note that m has a binary expansion of the form $x01^r0^{L-1}1$, where x is a (possibly empty) binary string. Since $m \geq 2^3 \cdot 1 + 1 = 9$, we have that $r \geq 1$. Let N be the number of 1's in x. The binary expansion of $m + 2^L + 2$ can be expressed as $x10^{r+L-2}11$, which has $N + 3$ 1's. Similarly, we obtain the following table:

i	Binary Expansion of i	Number of 1's in Binary Expansion of i
$m + 2^L + 2$	$x10^{r+L-2}11$	$N + 3$
$m + 2^L + 3$	$x10^{r+L-3}100$	$N + 2$
$m + 2^{L+1} + 2$	$x10^{r-1}10^{L-2}11$	$N + 4$
$m + 2^{L+1} + 3$	$x10^{r-1}10^{L-3}100$	$N + 3$

Recall that the parity of $\mathbf{t}_i$ is the same as the parity of the number of 1's in the binary expansion of $i - 1$. It follows that $\mathbf{t}_{m+2^L+3}\mathbf{t}_{m+2^L+4} = 01$ if N is odd and $\mathbf{t}_{m+2^{L+1}+3}\mathbf{t}_{m+2^{L+1}+4} = 01$ if N is even. Observe that $\mathbf{t}_{2^L+3}\mathbf{t}_{2^L+4} = \mathbf{t}_{2^{L+1}+3}\mathbf{t}_{2^{L+1}+4} = 01$. Therefore, setting $s = 2^L + 3$ yields a contradiction to Lemma 3.3 if N is odd, and setting $s = 2^{L+1} + 3$ yields the desired contradiction if N is even. □

Remark 3.6. The proof of Lemma 3.5 closely follows that of [8, Lemma 15]. Note, however, that in Defant's proof of [8, Lemma 15], he mistakenly claims that $\mathbf{t}_{2^L+3}\mathbf{t}_{2^L+4} = \mathbf{t}_{2^{L+1}+3}\mathbf{t}_{2^{L+1}+4} = 10$, rather than $\mathbf{t}_{2^L+3}\mathbf{t}_{2^L+4} = \mathbf{t}_{2^{L+1}+3}\mathbf{t}_{2^{L+1}+4} = 01$. Setting $j = 0$ in the above proof yields a correct proof of [8, Lemma 15].

Lemma 3.7. *Let $j \in \mathbb{Z}^{\geq 0}$. Suppose $m = 2^L h + 1$, where L and h are integers with $L \geq 3$ and h odd. Let $\ell = \lceil \log_2(m + j) \rceil$. If n is an integer such that $2 \leq n \leq 2^{L-1}$, $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$, and $m + j \leq \left(1 - \frac{1}{2n+2}\right) 2^\ell$, then*

$$\mathfrak{K}_j(m) \leq 2^{\ell+1} - \frac{2^{\ell+1}(n-1) + j}{m}.$$

Proof. For any m satisfying the hypotheses of the lemma, we have $\mathbf{t}_{m-2n}\mathbf{t}_{m-2n+1}\mathbf{t}_{m-2n+2} = \mathbf{t}_{2m-2n}\mathbf{t}_{2m-2n+1}\mathbf{t}_{2m-2n+2}$ [8, Lemma 16]. Consequently,

$$\begin{aligned}\langle (m-2n-1)2^\ell + 1, (m-2n+2)2^\ell \rangle &= \mu^\ell(\mathbf{t}_{m-2n}\mathbf{t}_{m-2n+1}\mathbf{t}_{m-2n+2}) \\ &= \mu^\ell(\mathbf{t}_{2m-2n}\mathbf{t}_{2m-2n+1}\mathbf{t}_{2m-2n+2}) = \langle (2m-2n-1)2^\ell + 1, (2m-2n+2)2^\ell \rangle.\end{aligned}$$

We want to show that there is an integer $r \leq 2^\ell - 1$ such that

$$(m-2n-1)2^\ell \leq (2^\ell - r - 1)m + j < (2^\ell - r)m + j < (m-2n+2)2^\ell. \tag{7}$$

To this end, note that

$$(m-2n+2)2^\ell - (m-2n-1)2^\ell = 3 \cdot 2^\ell \geq 3(m+j) \geq 3m$$

and that

$$((2^\ell - r)m + j) - ((2^\ell - r - 1)m + j) = m.$$

It follows that there exists $r \in \mathbb{Z}$ satisfying (7). We now verify that r can always be chosen such that $r \le 2^\ell - 1$. Our choice of r is forced to be largest when $m - 2n$ is smallest. Observe that

$$m - 2n - 1 = 2^L h - 2n \ge 2^L h - 2^L = 2^L(h-1) \ge 0.$$

Indeed, (7) is satisfied by $r = 2^\ell - 1$ when $m - 2n - 1 = 0$:

$$0 = (m-2n-1)2^\ell \le j = (2^\ell - r - 1)m + j < m + j = (2^\ell - r)m + j < 3\cdot 2^\ell = (m-2n+2)2^\ell.$$

Therefore, for some integer $r \le 2^\ell - 1$, there exist words w and z (with z nonempty) such that

$$(8)\quad \langle (m-2n-1)2^\ell + 1, (m-2n+2)2^\ell \rangle = w\langle (2^\ell - r - 1)m + j + 1, (2^\ell - r)m + j\rangle z.$$

Adding $2^\ell m$ to each index in (8) gives that there exist nonempty words w' and z' such that

$$(9)\quad \langle (2m-2n-1)2^\ell + 1, (2m-2n+2)2^\ell \rangle = w'\langle (2^{\ell+1} - r - 1)m + j + 1, (2^{\ell+1} - r)m + j\rangle z'.$$

Note that $|w'| = -rm - m + j + 2^{\ell+1}m + 2^\ell = |w|$. Therefore, (8) and (9) give that

$$\langle (2^\ell - r - 1)m + j + 1, (2^\ell - r)m + j\rangle = \langle (2^{\ell+1} - r - 1)m + j + 1, (2^{\ell+1} - r)m + j\rangle.$$

Noting from (7) that $r > \dfrac{2^{\ell+1}(n-1) + j}{m}$, we therefore have

$$\mathfrak{K}_j(m) \le 2^{\ell+1} - r \le 2^{\ell+1} - \frac{2^{\ell+1}(n-1) + j}{m},$$

as desired. □

We now address the case in which $m \equiv 29 \pmod{32}$.

Lemma 3.8. *Let m be a positive integer satisfying $m \equiv 29 \pmod{32}$. Let $j \in \mathbb{Z}^{\ge 0}$, and let $\ell = \lceil \log_2(m+j) \rceil$. We have*

$$\mathfrak{K}_j(m) < 2^{\ell+1} + \frac{20 \cdot 2^\ell - j}{m}.$$

Proof. Suppose $m = 32n - 3$. Let N be the number of 1's in the binary expansion of n. It is straightforward to verify that the binary expansion of $m + 17 = 32n + 14$ has $N + 3$ 1's. Similarly, we obtain the following table:

i	Number of 1's in Binary Expansion of i	$\mathbf{t}_{i+1}$
$m+17$	$N+3$	1
$m+18$	$N+4$	0
$m+19$	$N+1$	1
$2m+17$	$N+3$	1
$2m+18$	$N+2$	0
$2m+19$	$N+3$	1

Consequently, we have that $\mathbf{t}_{m+18}\mathbf{t}_{m+19}\mathbf{t}_{m+20} = \mathbf{t}_{2m+18}\mathbf{t}_{2m+19}\mathbf{t}_{2m+20}$. It follows that

$$\begin{aligned}\langle (m+17)2^\ell + 1, (m+20)2^\ell \rangle &= \mu^\ell(\mathbf{t}_{m+18}\mathbf{t}_{m+19}\mathbf{t}_{m+20}) \\ &= \mu^\ell(\mathbf{t}_{2m+18}\mathbf{t}_{2m+19}\mathbf{t}_{2m+20}) = \langle (2m+17)2^\ell + 1, (2m+20)2^\ell \rangle.\end{aligned}$$

Applying Lemma 3.1 with $s = 18$ and $a = 2$ gives that there exists $r \in \mathbb{Z}^{\geq 0}$ such that

$$\langle 2^\ell \cdot 17 + 1, 2^\ell \cdot 20 \rangle = w \langle rm + j + 1, (r+1)m + j \rangle z \tag{10}$$

for some words w and z (with z nonempty). Adding $2^\ell m$ to each index in (10) implies that

$$\langle 2^\ell(m+17) + 1, 2^\ell(m+20) \rangle = w' \langle (r + 2^\ell)m + j + 1, (r + 2^\ell + 1)m + j \rangle z' \tag{11}$$

for some words w' and z' (with z' possibly empty). Similarly, adding $2^{\ell+1}m$ to each index in equation (10) gives that there exist words w'' and z'' (with z'' nonempty) for which

$$\langle 2^\ell(2m+17)+1, 2^\ell(2m+20) \rangle = w'' \langle (r+2^{\ell+1})m+j+1, (r+2^{\ell+1}+1)m+j \rangle z''. \tag{12}$$

Observe that $|w''| = rm + j - 17 \cdot 2^\ell = |w'|$. Therefore, (11) and (12) imply

$$\langle (r+2^\ell)m + j + 1, (r + 2^\ell + 1)m + j \rangle = \langle (r + 2^{\ell+1})m + j + 1, (r + 2^{\ell+1} + 1)m + j \rangle.$$

Noting from (10) that $r + 1 < \dfrac{20 \cdot 2^\ell - j}{m}$, we get

$$\mathfrak{K}_j(m) \leq r + 2^{\ell+1} + 1 < 2^{\ell+1} + \frac{20 \cdot 2^\ell - j}{m},$$

as desired. □

Remark 3.9. We make note of an error in Defant's proof of an upper bound for $\mathfrak{K}_0(m)$ in the case $m \equiv 29 \pmod{32}$. In Defant's proof of [8, Lemma 14], he claims that

$$\bigcup_{r=9}^{17} \left(\frac{17}{2r}, \frac{10}{r+1} \right) = \left(\frac{1}{2}, 1 \right), \tag{13}$$

which implies the existence of some $r \in \{9, 10, \ldots, 17\}$ such that $\dfrac{17}{2r} < \dfrac{m}{2^\ell} < \dfrac{10}{r+1}$, where $\ell = \lceil \log_2 m \rceil$. However, (13) is in fact false. This mistake can be highlighted by observing that for $m = 32 \cdot 15 - 3 = 477$, there does not exist $r \in \{9, 10, \ldots, 17\}$ satisfying the desired inequality. Fortunately, setting $j = 0$ in Lemma 3.8 gives the bound $\mathfrak{K}_0(m) < 2^{\ell+1} + \dfrac{20 \cdot 2^\ell}{m}$, which is only slightly worse than Defant's intended bound of $\mathfrak{K}_0(m) \leq 2^\ell + 18$. This worsens Defant's lower bound for $\liminf\limits_{k\to\infty}(\gamma_0(k)/k)$

from 1/2 to 1/4, and his lower bound for $\limsup_{k\to\infty}(\gamma_0(k)/k)$ from 1 to 1/2. However, Narayanan [12] proves $\liminf_{k\to\infty}(\gamma_0(k)/k) \geq 3/4$ and $\limsup_{k\to\infty}(\gamma_0(k)/k) = 3/2$, so we still know Defant's claimed lower bounds to be true.

Finally, we consider the case in which m is an odd positive integer with $m \not\equiv 1 \pmod 8$ and $m \not\equiv 29 \pmod{32}$. In this case, we can apply Defant's proof of [8, Lemma 14] almost exactly. For the reader's convenience, we include a slightly augmented outline of this proof as the proof of Lemma 3.10; for more details, see [8, Lemma 14].

Lemma 3.10. *Let m be an odd positive integer with $m \not\equiv 1 \pmod 8$ and $m \not\equiv 29 \pmod{32}$. Let $j \in \mathbb{Z}^{\geq 0}$, and let $\ell = \lceil \log_2(m+j) \rceil$. We have*

$$\mathfrak{K}_j(m) < 2^\ell + \frac{37 \cdot 2^\ell - j}{m}.$$

Proof. Suppose for the sake of contradiction that $\mathfrak{K}_j(m) \geq 2^\ell + \dfrac{37 \cdot 2^\ell - j}{m}$. When $m \equiv 3 \pmod 4$ or $m \equiv 5 \pmod 8$ (while $m \not\equiv 29 \pmod{32}$), we will obtain a contradiction to Lemma 3.3 by exhibiting a positive integer $s \leq 36$ satisfying $\mathbf{t}_s\mathbf{t}_{s+1} = \mathbf{t}_{m+s}\mathbf{t}_{m+s+1}$.

Assume first that $m \equiv 3 \pmod 4$. In this case, $\mu^2(\mathbf{t}_{(m+5)/4}) = \langle m+2, m+5\rangle$, so we have either $\langle m+2, m+5\rangle = 0110$ or $\langle m+2, m+5\rangle = 1001$. Since $\mathfrak{K}_j(m) > 2^\ell + 1$, we have by Lemma 3.2 that $\mathbf{t}_{m+2} = 1$. It follows that $\langle m+2, m+5\rangle = 1001$. In particular, $\mathbf{t}_{m+4}\mathbf{t}_{m+5} = 01 = \mathbf{t}_4\mathbf{t}_5$. Therefore, setting $s = 4$ yields a contradiction to Lemma 3.2.

Assume next that $m \equiv 5 \pmod 8$ while $m \not\equiv 29 \pmod{32}$. Note that m has a binary expansion of the form $x01^r01$, where x is a (possibly empty) binary string. Since $m \equiv 5 \pmod 8$ and $m \not\equiv 29 \pmod{32}$, we have that $1 \leq r \leq 2$. Lemma 3.2 gives that $\mathbf{t}_{m+1} = 1$, meaning the number of 1's in the binary expansion of m is odd. It follows that the parity of the number of 1's in x is the same as the parity of r.

Suppose $r = 1$. Defant shows that in this case, $\mathbf{t}_{m+4}\mathbf{t}_{m+5} = 01 = \mathbf{t}_4\mathbf{t}_5$, so we may again set $s = 4$ to yield a contradiction to Lemma 3.2.

Suppose that $r = 2$ and that x ends in a 0. In this case, Defant argues that $\mathbf{t}_{m+20}\mathbf{t}_{m+21} = 10 = \mathbf{t}_{20}\mathbf{t}_{21}$, so we may set $s = 20$ to contradict Lemma 3.2.

Finally, suppose that $r = 2$ and that x ends in a 1. Let us write $x = x'01^{r'}$, where x' is a (possibly empty) binary string. Defant shows we can put $s = 20$ if r' is even and $s = 36$ if r' is odd to yield contradictions to Lemma 3.2. □

The following two lemmas use the preceding results to establish a single upper bound for $\mathfrak{K}_j(m)$ for any integer $m \geq 2$.

Lemma 3.11. *Let $j \in \mathbb{Z}^{\geq 0}$, and suppose $m = 2^L h + 1$, where L and h are integers with $L \geq 3$ and h odd. Let $\ell = \lceil \log_2(m+j) \rceil$. Then*

$$\mathfrak{K}_j(m) \leq 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}.$$

Proof. First, assume that $m + j > \left(1 - \frac{1}{2^L - 4}\right) 2^\ell$. Observe that $2^\ell - 2^L h = 2^\ell - m + 1$. Since $L < \ell$, we have that 2^L divides $2^\ell - 2^L h$, which further gives that 2^L divides $2^\ell - m + 1$. Since $2^\ell - m + 1 > 0$, this gives that

$$2^L \leq 2^\ell - m + 1 < 2^\ell - \left(2^\ell - \frac{2^\ell}{2^L - 4} - j\right) + 1 = \frac{2^\ell}{2^L - 4} + j + 1.$$

This implies that $2^{2L} - 4 \cdot 2^L < 2^\ell + j(2^L - 4) + 2^L - 4$. Rearranging and dividing by 2^L gives the first inequality of

$$2^L < 2^{\ell - L} + (j+5) - 4(j+1)2^{-L} < 2^{\ell - L + 2} + 2^\ell - m - 4(j+1)2^{-L}; \tag{14}$$

the second inequality is straightforward to verify. From Lemma 3.5, we have that $\mathfrak{K}_j(m) < 2^\ell + \frac{2^\ell(2^{L+1} + 4) - j}{m}$. Incorporating (14), we get

$$\begin{aligned}
2^\ell(2^{L+1} + 4) - j &= 2^{\ell+1} \cdot 2^L + 2 \cdot 2^{\ell+1} - j \\
&< 2^{\ell+1}(2^{\ell - L + 2} + 2^\ell - m - 4(j+1)2^{-L}) + 8 \cdot 2^{\ell-1} - j \\
&\leq (2^\ell - 1)2^{\ell - L + 3} + (2^{\ell+1} + 8)2^{\ell-1} + (2^{\ell+1} - 2^{\ell - L + 3} - 1)j \\
&\leq (2^{\ell+1} + 3)2^\ell + (2^{\ell+1} - 15)j,
\end{aligned}$$

where, in the last step, we have used that $\ell = \lceil \log_2(m+j) \rceil \geq L + 1$ and that $L \geq 3$. It follows that

$$\begin{aligned}
\mathfrak{K}_j(m) &< 2^\ell + \frac{(2^{\ell+1} + 3)2^\ell + (2^{\ell+1} - 15)j}{m} \\
&\leq 2^\ell + \frac{(2^{\ell+1} + 3)2^\ell + (2^{\ell+1} - 15)j}{2^{\ell-1} - j} \leq 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}.
\end{aligned}$$

Next, assume that $m + j \leq \left(1 - \frac{1}{2^L - 4}\right) 2^\ell$ and $L \geq 4$. Let n be the largest integer such that $m - n \equiv 2 \pmod 4$ and $n \leq 2^{L-1}$. Since $n \geq 2^{L-1} - 3$, we have that $m + j \leq \left(1 - \frac{1}{2n+2}\right) 2^\ell$. By the condition $m - n \equiv 2 \pmod 4$, we have $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$. We can therefore apply Lemma 3.7, which gives

$$\mathfrak{K}_j(m) \leq 2^{\ell+1} - \frac{2^{\ell+1}(n-1) + j}{m} \leq 2^{\ell+1} - \frac{2^{\ell+1}(2^{L-1} - 4)}{2^\ell - j} \leq 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}.$$

Finally, suppose $L = 3$. By Lemma 3.5,

$$\mathfrak{K}_j(m) < 2^\ell + \frac{20 \cdot 2^\ell - j}{m} \leq 2^\ell + \frac{20 \cdot 2^\ell - j}{2^{\ell-1} - j} < 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}. \qquad \square$$

Lemma 3.12. *Let $j, m \in \mathbb{Z}^{\geq 0}$ with $m \geq 2$ and $m \not\equiv 1 \pmod 8$. Let $\ell = \lceil \log_2(m + j) \rceil$. Then*

$$\mathfrak{K}_j(m) \leq 2^\ell + \frac{2^{\ell+1} \cdot \max\{2^\ell + 2 + j, 20\}}{2^{\ell-1} - j}.$$

Proof. If $m \equiv 0 \pmod 2$, we have by Lemma 3.4 that

$$\mathfrak{K}_j(m) < 2^{\ell+1} + \frac{2^{\ell+1} - j}{m} \leq 2^{\ell+1} + \frac{2^{\ell+1} - j}{2^{\ell-1} - j} < 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}.$$

If $m \equiv 29 \pmod{32}$, we have by Lemma 3.8 that

$$\mathfrak{K}_j(m) < 2^{\ell+1} + \frac{20 \cdot 2^\ell - j}{m} \leq 2^{\ell+1} + \frac{20 \cdot 2^\ell - j}{2^{\ell-1} - j} < 2^\ell + \frac{2^{\ell+1}(2^\ell + 2 + j)}{2^{\ell-1} - j}.$$

Finally, if m is an odd positive integer with $m \not\equiv 1 \pmod 8$ and $m \not\equiv 29 \pmod{32}$, we have by Lemma 3.10 that

$$\mathfrak{K}_j(m) < 2^\ell + \frac{37 \cdot 2^\ell - j}{m} < 2^\ell + \frac{37 \cdot 2^\ell - j}{2^{\ell-1} - j} < 2^\ell + \frac{20 \cdot 2^{\ell+1}}{2^{\ell-1} - j}. \qquad \square$$

We are now in a position to prove the lower bounds for $\liminf_{k\to\infty}(\gamma_j(k)/k)$ and $\limsup_{k\to\infty}(\gamma_j(k)/k)$.

Theorem 3.13. *For any nonnegative integer j,*

$$\liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \geq \frac{1}{10} \quad \text{and} \quad \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \geq \frac{1}{5}.$$

Proof. Fix $j \in \mathbb{Z}^{\geq 0}$. For each positive integer ℓ, define $g_j(\ell) = 2^\ell + \dfrac{2^{\ell+1} \cdot \max\{2^\ell + 2 + j, 20\}}{2^{\ell-1} - j}$. Choose an arbitrary $k \in \mathbb{Z}^+$ and let $\ell = \lceil \log_2(\gamma_j(k) + j) \rceil$. By definition of γ_j, we have that $k < \mathfrak{K}_j(\gamma_j(k))$. Applying Lemmas 3.11 and 3.12 gives $\dfrac{\gamma_j(k)}{k} > \dfrac{\gamma_j(k)}{g_j(\ell)} > \dfrac{2^{\ell-1} - j}{g_j(\ell)}$. Therefore, $\liminf_{k\to\infty} \dfrac{\gamma_j(k)}{k} \geq \lim_{\ell\to\infty} \dfrac{2^{\ell-1} - j}{g_j(\ell)} = \dfrac{1}{10}$.

By Lemmas 3.11 and 3.12, we have that $\mathfrak{K}_j(m) < \lfloor g_j(\ell) \rfloor + 1$ for all positive integers $m < 2^\ell - j$. Therefore, by the definition of γ_j, we have that $\gamma_j(\lfloor g_j(\ell) \rfloor + 1) \geq 2^\ell - j + 1$. Consequently,

$$\limsup_{k\to\infty} \frac{\gamma_j(\ell)}{k} \geq \limsup_{\ell\to\infty} \frac{\gamma_j(\lfloor g_j(\ell) \rfloor + 1)}{\lfloor g_j(\ell) \rfloor + 1} \geq \lim_{\ell\to\infty} \frac{2^\ell - j + 1}{g_j(\ell) + 1} = \frac{1}{5}. \qquad \square$$

3.2. **Upper Bounds for $\gamma_j(k)/k$.** In this subsection we establish upper bounds for $\liminf_{k\to\infty}(\gamma_j(k)/k)$ and $\limsup_{k\to\infty}(\gamma_j(k)/k)$. We start by stating a result of Defant.

Proposition 3.14 ([8], Proposition 6)**.** *Let $m \geq 2$ be an integer, and let $\delta(m) = \lceil \log_2(m/3) \rceil$. If y and v are words such that yvy is a factor of $\mathbf{t}$ and $|y| = m$, then $2^{\delta(m)}$ divides $|yv|$.*

We proceed with a lemma and theorem whose proofs closely follow those of [8, Lemma 19] and [8, Theorem 20], respectively.

Lemma 3.15. *For each integer $\ell \geq 3$ and any nonnegative integer j, we have*

$$\mathfrak{K}_j(3 \cdot 2^{\ell-2} + 1) > \frac{5 \cdot 2^{2\ell-3} - j}{3 \cdot 2^{\ell-2} + 1} \quad \text{and} \quad \mathfrak{K}_j(2^{\ell-1} + 3) > \frac{2^{2\ell-2} - j}{m'}.$$

Proof. Fix $\ell \geq 3$ and $j \in \mathbb{Z}^{\geq 0}$. Let $m = 3 \cdot 2^{\ell-2} + 1$ and $m' = 2^{\ell-1} + 3$. By the definitions of $\mathfrak{K}_j(m)$ and $\mathfrak{K}_j(m')$, there exist nonnegative integers $r < \mathfrak{K}_j(m) - 1$ and $r' < \mathfrak{K}_j(m') - 1$ such that

$$\langle rm + j + 1, (r+1)m + j \rangle = \langle (\mathfrak{K}_j(m) - 1)m + j + 1, \mathfrak{K}_j(m)m + j \rangle$$

and

$$\langle r'm' + j + 1, (r'+1)m' + j \rangle = \langle (\mathfrak{K}_j(m') - 1)m' + j + 1, \mathfrak{K}_j(m')m' + j \rangle.$$

By Proposition 3.14, $2^{\ell-1}$ divides $(\mathfrak{K}_j(m) - 1)m - rm$ and $2^{\ell-2}$ divides $(\mathfrak{K}_j(m') - 1)m' - r'm'$. Because m and m' are odd, we have that $2^{\ell-1}$ divides $\mathfrak{K}_j(m) - r - 1$ and $2^{\ell-2}$ divides $\mathfrak{K}_j(m') - r' - 1$. If $\mathfrak{K}_j(m) - r - 1 \geq 2^\ell$, then we have the desired inequality $\mathfrak{K}_j(m) > \dfrac{5 \cdot 2^{2\ell-3} - j}{3 \cdot 2^{\ell-2} + 1}$. We may therefore assume that $\mathfrak{K}_j(m) = r + 2^{\ell-1} + 1$. Similarly, we may assume that $\mathfrak{K}_j(m') = r' + 2^{\ell-2} + 1$.

Assume for the sake of contradiction that $\mathfrak{K}_j(m) \leq \dfrac{5 \cdot 2^{2\ell-3} - j}{m}$. Let $u = \langle rm + j + 1, (r+1)m + j \rangle$ and $v = \langle (\mathfrak{K}_j(m) - 1)m + j + 1, \mathfrak{K}_j(m)m + j \rangle$. It is straightforward to verify that

$$3 \cdot 2^{2\ell-3} < (\mathfrak{K}_j(m) - 1)m + j < \mathfrak{K}_j(m)m + j \leq 5 \cdot 2^{2\ell-3}.$$

Therefore, we have

$$\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_4\mathbf{t}_5) = \langle 3 \cdot 2^{2\ell-3} + 1, 5 \cdot 2^{2\ell-3} \rangle = wvz$$

for some words w and z. Observe that $|w| = ((\mathfrak{K}_j(m) - 1)m + j + 1) - 3 \cdot 2^{2\ell-3} = rm + 2^{\ell-1} + j$. Since $\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_1\mathbf{t}_2) = \langle 1, 2^{2\ell-3} \rangle$, we have $v = \langle rm + 2^{\ell-1} + j + 1, (r+1)m + 2^{\ell-1} + j \rangle$. Now, set $a = rm + j + 1$ and $b = rm + 2^{\ell-1} + j + 1$, and note that $a < b \leq a + m$. Recalling that $\mathbf{t}$ is overlap-free, this implies that $u \neq v$, a contradiction.

Assume now that $\mathfrak{K}_j(m') \le \dfrac{2^{2\ell-2}-j}{m'}$. Let $u' = \langle r'm'+j+1, (r'+1)m'+j\rangle$ and $v' = \langle(\mathfrak{K}_j(m')-1)m'+j+1, \mathfrak{K}_j(m')m'\rangle$. Let $q = \left\lceil \dfrac{r'm'+j+1}{2^{\ell-2}} \right\rceil$ and $H = \min\left\{(r'+1)m', (q+2)2^{\ell-2}+j\right\}$. Additionally, let $U = \langle r'm'+j+1, H+j\rangle$ and $V = \langle (r'+2^{\ell-2})m'+j+1, H+2^{\ell-2}m'+j\rangle$. Note that the word U is the prefix of u' of length $H-r'm'$. Recalling that $\mathfrak{K}_j(m') = r'+2^{\ell-2}+1$, we see that V is the prefix of v' of length $H-r'm'$. Since $u'=v'$, it follows that $U=V$.

Now, we claim that there are words w' and z' such that

$$\mu^{\ell-2}(\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}) = \langle (q-1)2^{\ell-2}+1, (q+2)2^{\ell-2}\rangle = w'Uz'.$$

This can be easily verified by checking that $(q-1)2^{\ell-2} \le r'm'+j < H+j \le (q+2)2^{\ell-2}$. Similarly, there are words w'' and z'' such that

$$\mu^{\ell-2}(\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}) = \langle (q+m'-1)2^{\ell-2}+1, (q+m'+2)2^{\ell-2}\rangle = w''Vz''.$$

Note that

$$0 \le |w'| = |w''| = r'm'+j-(q-1)2^{\ell-2} \le r'm'+j-\left(\frac{r'm'+j+1}{2^{\ell-2}}-1\right)2^{\ell-2} < 2^{\ell-2},$$

meaning w' is a prefix of $\mu^{\ell-2}(\mathbf{t}_q)$ and w'' is a prefix of $\mu^{\ell-2}(\mathbf{t}_{q+m'+1})$. Therefore, the suffix of $\mu^{\ell-2}(\mathbf{t}_q)$ of length $2^{\ell-2}-|w'|$ is a prefix of U and the suffix of $\mu^{\ell-2}(\mathbf{t}_{q+m'})$ of length $2^{\ell-2}-|w''|$ is a prefix of V. Since $|w'|=|w''|$ and $U=V$, it follows that $\mathbf{t}_q = \mathbf{t}_{q+m'}$.

Note also that $|z'| = |z''| = (q+2)2^{\ell-2}-(H+j)$. We will show that $H+2^{\ell-2}m+j+1-(q+m'+1)2^{\ell-2} > 0$, which will show that z'' is a suffix of $\mu^{\ell-2}(\mathbf{t}_{q+m'+2})$. Observe that

$$\begin{aligned} H+2^{\ell-2}m'+j+1-(q+m'+1)2^{\ell-2} &= H+j+1-q2^{\ell-2}-2^{\ell-2} \\ &> H+j+1-\left(\frac{r'm'+j+1}{2^{\ell-2}}+1\right)2^{\ell-2}-2^{\ell-2} \\ &= H-r'm'-2^{\ell-1}. \end{aligned}$$

If $H = r'm'+m'$, then $H = r'm'+2^{\ell-1}+3 > r'm'+2^{\ell-1}$, giving $H-r'm'-2^{\ell-1} > 0$. Alternatively, if $H = (q+2)2^{\ell-2}-j$, then we have

$$(q+2)2^{\ell-2}-j \ge \left(\frac{r'm'+j+1}{2^{\ell-2}}+2\right)2^{\ell-2}-j = r'm'+2^{\ell-1}+1 > r'm'+2^{\ell-1},$$

and again $H-r'm'-2^{\ell-1} > 0$. It follows that $\mathbf{t}_{q+2} = \mathbf{t}_{q+m'+2}$. Similarly, $\mathbf{t}_{q+1} = \mathbf{t}_{q+m'+1}$.

$\langle (q-1)2^{\ell-2}+1,(q+2)2^{\ell-2}\rangle$				$\langle (q+m'-1)2^{\ell-2}+1,(q+m'+2)2^{\ell-2}\rangle$		
$\mu^{\ell-2}(\mathbf{t}_q)$	$\mu^{\ell-2}(\mathbf{t}_{q+1})$	$\mu^{\ell-2}(\mathbf{t}_{q+2})$		$\mu^{\ell-2}(\mathbf{t}_{q+m'})$	$\mu^{\ell-2}(\mathbf{t}_{q+m'+1})$	$\mu^{\ell-2}(\mathbf{t}_{q+m'+2})$
w'	U	z'		w''	V	z''

FIGURE 1. An illustration of the proof of Lemma 3.15 from [8].

Now,

$$r' = \mathfrak{K}_j(m') - 2^{\ell-2} - 1 \leq \frac{2^{2\ell-2}-j}{m'} - 2^{\ell-2} - 1 = \frac{2^{2\ell-3}-5\cdot 2^{\ell-2}-j-3}{m'}.$$

It follows that $r'm'+j+1 \leq 2^{2\ell-3}-5\cdot 2^{\ell-2}-2$, which gives that $\dfrac{r'm'+j+1}{2^{\ell-2}} \leq 2^{\ell-1}-5$. Therefore, $q+4 < 2^{\ell-1}$. Consequently, for each $s \in \{0,1,2\}$, the binary expansion of $q+m'+s-1$ has exactly one more 1 than the binary expansion of $q+s+2$. Thus,

$$\mathbf{t}_{q+3}\mathbf{t}_{q+4}\mathbf{t}_{q+5} = \overline{\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}} = \overline{\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}}.$$

However, using that $\mathbf{t}$ is cube-free, it is easy to verify that whenever X is a word of length 3, $X\overline{X}$ is not a factor of $\mathbf{t}$. Setting $X = \overline{\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}}$ therefore yields a contradiction. □

Theorem 3.16. *For any nonnegative integer* j*,*

$$\liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{9}{10} \qquad \text{and} \qquad \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{3}{2}.$$

Proof. Fix $j \in \mathbb{Z}^{\geq 0}$. For each positive integer ℓ, let $f_j(\ell) = \left\lfloor \dfrac{5\cdot 2^{2\ell-3}-j}{3\cdot 2^{\ell-2}+1} \right\rfloor$ and $h_j(\ell) = \left\lfloor \dfrac{2^{2\ell-2}-j}{2^{\ell-1}+3} \right\rfloor$. It is straightforward to verify that $h_j(\ell) < f_j(\ell) \leq h_j(\ell+1)$ for all $\ell \geq 3$. By Lemma 3.15, we have that $\mathfrak{K}_j(3\cdot 2^{\ell-2}+1) > f_j(\ell)$. As a result, the j-fix of $\mathbf{t}$ of length $(3\cdot 2^{\ell-2}+1)f_j(\ell)$ is an $f_j(\ell)$-anti-power, meaning $\gamma_j(f_j(\ell)) \leq 3\cdot 2^{\ell-2}+1$. Consequently,

$$\liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \liminf_{\ell\to\infty} \frac{\gamma_j(f_j(\ell))}{f_j(\ell)} \leq \liminf_{\ell\to\infty} \frac{3\cdot 2^{\ell-2}+1}{f_j(\ell)} = \frac{9}{10}.$$

Fix an integer $k \geq 3$. Suppose that $h_j(\ell) < k \leq f_j(\ell)$ for some integer $\ell \geq 3$. In this case, the j-fix of $\mathbf{t}$ of length $(3\cdot 2^{\ell-2}+1)f_j(\ell)$ is an $f_j(\ell)$-anti-power. It follows that $\gamma_j(k) \leq 3 \cdot 2^{\ell-2} + 1$, meaning

$$\frac{\gamma_j(k)}{k} < \frac{3 \cdot 2^{\ell-2} + 1}{h_j(\ell)}.$$

Alternatively, suppose that $f_j(\ell) < k \leq h_j(\ell+1)$ for some $\ell \geq 3$. In this case, Lemma 3.15 gives that the j-fix of $\mathbf{t}$ of length $(2^\ell + 3)h_j(\ell + 1)$ is an $h_j(\ell + 1)$-anti-power, meaning

$$\frac{\gamma_j(k)}{k} < \frac{2^\ell + 3}{f_j(\ell)}.$$

We can now combine the above cases to see that

$$\limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \limsup_{\ell\to\infty} \left(\max\left\{\frac{3 \cdot 2^{\ell-2} + 1}{h_j(\ell)}, \frac{2^\ell + 3}{f_j(\ell)}\right\}\right) = \max\left\{\frac{3}{2}, \frac{6}{5}\right\} = \frac{3}{2}.$$

□

4. Asymptotics for $\Gamma_j(k)$

Having established asymptotic bounds showing that $\gamma_j(k)$ grows linearly in k, we now turn our attention to $\Gamma_j(k)$. In this section, we prove that $\liminf_{k\to\infty}(\Gamma_j(k)/k) = 3/2$ and $\limsup_{k\to\infty}(\Gamma_j(k)/k) = 3$. We start by motivating our definition of $\Gamma_j(k)$.

Recall that we have defined $\Gamma_j(k) := \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(k))$. Also recall that Defant's motivation for defining $\Gamma_0(k) := \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_0(k))$ is the property that $m \in AP_0(\mathbf{t}, k)$ if and only if $2m \in AP_0(\mathbf{t}, k)$, meaning that the only interesting elements of $AP_0(\mathbf{t}, k)$ are those that are odd. However, as previously noted, it is not necessarily the case for nonzero j that $m \in AP_j(\mathbf{t}, k)$ if and only if $2m \in AP_j(\mathbf{t}, k)$. As such, it is not initially clear that we are motivated in generalizing Defant's definition of $\Gamma_0(k)$ in the way we have. In other words, if it is possible for even elements of $AP_j(\mathbf{t}, k)$ to be interesting, why would we consider only the odd elements? The following proposition demonstrates a drawback of considering all even elements of $AP_j(\mathbf{t}, k)$.

Proposition 4.1. *For $k \geq 3$, the set $2\mathbb{Z}^+ \setminus (AP_0(\mathbf{t}, k) \cap 2\mathbb{Z}^+)$ is unbounded.*

Proof. Since $\mathbf{t}_1\mathbf{t}_2 \cdots \mathbf{t}_9 = 011010011$ has two occurrences of 011, we have that $3 \in \mathbb{Z}^+ \setminus AP_0(\mathbf{t}, k)$ for all $k \geq 3$. Recall that $m \in AP_0(\mathbf{t}, k)$ if and only if $2m \in AP_0(\mathbf{t}, k)$. Therefore, $3 \cdot 2^L \in 2\mathbb{Z}^+ \setminus (AP_0(\mathbf{t}, k) \cap 2\mathbb{Z}^+)$ for all $L \in \mathbb{Z}^+$. The proposition follows. □

As a consequence of Proposition 4.1, if we were to include even numbers by defining $\Gamma_j(k) := \sup(\mathbb{Z}^+ \setminus AP_j(\mathbf{t}, k))$, we would have that $\Gamma_0(k) = \infty$ for $k \geq 3$, which is contrary to the result we are trying to generalize (namely, that $\Gamma_0(k)$ grows linearly in k). As further motivation for our definition of $\Gamma_j(k)$, we make the following conjecture.

Conjecture 4.2. *For any fixed $j, k \in \mathbb{Z}^{\geq 0}$ with $k \geq 3$, the statement*

$$m \in AP_j(\mathbf{t}, k) \iff 2m \in AP_j(\mathbf{t}, k)$$

holds for all but finitely many $m \in \mathbb{Z}^+$.

This conjecture is supported by numerical evidence. For instance, consider $j \in \{1, 2, 3\}$, $3 \leq k \leq 40$, and $1 \leq m \leq 1000$. Then for each pair (j, k), the expected number of values of m not satisfying $m \in AP_j(\mathbf{t}, k) \iff 2m \in AP_j(\mathbf{t}, k)$ is less than 0.5.

A proof of this conjecture would likely involve a characterization of exactly when $m \in AP_j(\mathbf{t}, k) \iff 2m \in AP_j(\mathbf{t}, k)$, which would tell us precisely which elements of $AP_j(\mathbf{t}, k)$ are interesting. For now, all we can say for certain is that the odd elements of $AP_j(\mathbf{t}, k)$ are interesting, so we move forward with our definition of $\Gamma_j(k)$. Let us begin by proving a Corollary to [8, Proposition 6] (stated above as Proposition 3.14).

Corollary 4.3. *Let $m, k \in \mathbb{Z}^+$, where $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(\mathbf{t}, k)$ and $k \geq 3$. Let $\delta(m) = \lceil \log_2(m/3) \rceil$. Then $k - 1 \geq 2^{\delta(m)}$.*

Proof. By the hypotheses of the corollary, we have that the j-fix of $\mathbf{t}$ of length km is not a k-anti-power. It follows that there exist integers n_1 and n_2 with $0 \leq n_1 < n_2 \leq k - 1$ such that

$$\langle n_1 m + j + 1, (n_1 + 1)m + j \rangle = \langle n_2 m + j + 1, (n_2 + 1)m + j \rangle.$$

Let $y = \langle n_1 m + j + 1, (n_1 + 1)m + j \rangle$ and $v = \langle (n_1 + 1)m + j + 1, n_2 m + j \rangle$. The word yvy is a factor of $\mathbf{t}$, and $|y| = m$. We can therefore apply [8, Proposition 6] to get that $2^{\delta(m)}$ divides $|yv| = (n_2 - n_1)m$. Since m is odd, $2^{\delta(m)}$ divides $n_2 - n_1$. It follows that $k - 1 \geq n_2 - n_1 \geq 2^{\delta(m)}$. □

We now present a technical lemma that will be useful for constructing identical pairs of subwords of the Thue-Morse word. These pairs of subwords will allow us to establish upper bounds on $\mathfrak{K}_j(m)$ for certain odd values of m. It will be useful to keep in mind that $\Gamma_j(k) \geq m$ whenever $k \geq \mathfrak{K}_j(m)$; this fact follows from Definitions 1.3 and 1.5.

Lemma 4.4. *Suppose that r, m, ℓ, h, p, q are nonnegative integers satisfying the following conditions:*

- $h < 2^{\ell-2}$
- $2 \leq m < 2^{\ell}$
- $rm = 2^{\ell+1}p + 2^{\ell-1} + h - j$
- $(r + 1)m \leq 2^{\ell+1}p + 5 \cdot 2^{\ell-2} - j$
- $(r + 2^{\ell-2})m = 2^{\ell+1}q + 3 \cdot 2^{\ell-2} + h - j$
- $\mathbf{t}_{p+1} \neq \mathbf{t}_{q+1}$

Then $\langle rm+j+1,(r+1)m+j\rangle = \langle (r+2^{\ell-2})m+1,(r+2^{\ell-2}+1)m\rangle$*, and* $\mathfrak{K}_j(m) \leq r+2^{\ell-2}+1$.

Proof. Define $u = \langle rm+j+1,(r+1)m+j\rangle$ and $v = \langle (r+2^{\ell-2})m+j+1,(r+2^{\ell-2}+1)m+j\rangle$. Assume $\mathbf{t}_{p+1}=0$; a similar argument holds of $\mathbf{t}_{p+1}=1$. Recall the definitions of A_n and B_n from Definition 1.1.

We will first show that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = xuy$ for some words x and y with $|x|=h$. To this end, note that

$$\langle 2^{\ell+1}p+1, 2^{\ell+1}(p+1)\rangle = \mu^{\ell+1}(\mathbf{t}_{p+1}) = \mu^{\ell+1}(0) = A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}.$$

Noting that $|A_{\ell-2}| = |B_{\ell-2}| = 2^{\ell-2}$, it suffices to show that

$$2^{\ell+1}p+2\cdot 2^{\ell-2}+1 \leq rm+j+1 < (r+1)m+j \leq 2^{\ell+1}p+5\cdot 2^{\ell-2}. \tag{15}$$

To prove the leftmost inequality of (15), we use the third condition to note that

$$(rm+j)-(2^{\ell+1}p+2^{\ell-1}) = (2^{\ell+1}p+2^{\ell-1}+h)-(2^{\ell+1}p+2^{\ell-1}) = h \geq 0.$$

The middle inequality of (15) follows from the second condition, while the rightmost follows from the fourth. It follows that for some words x and y we have $B_{\ell-2}A_{\ell-2}B_{\ell-2} = xuy$.

We will now show that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = x'vy'$ for some words x' and y' with $|x'|=h$. To this end, note that

$$\langle 2^{\ell+1}q+1, 2^{\ell+1}(q+1)\rangle = \mu^{\ell+1}(\mathbf{t}_{q+1}) = \mu^{\ell+1}(1) = B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2},$$

where we have used the final condition to see that $\mathbf{t}_{q+1}=1$. Recalling that $|A_{\ell-2}| = |B_{\ell-2}| = 2^{\ell-2}$, it suffices to show that

$$2^{\ell+1}q+3\cdot 2^{\ell-2} \leq (r+2^{\ell-2})m+j < (r+2^{\ell-2}+1)m+j < 2^{\ell+1}q+6\cdot 2^{\ell-2}. \tag{16}$$

The leftmost inequality of (16) follows from an application of the fifth condition:

$$((r+2^{\ell-2})m+j)-(2^{\ell+1}q+3\cdot 2^{\ell-2}) = (2^{\ell+1}q+3\cdot 2^{\ell-2}+h)-(2^{\ell+1}q+3\cdot 2^{\ell-2}) = h \geq 0.$$

As before, the middle inequality in (16) follows from the second condition. For the rightmost inequality, note that

$$(r+2^{\ell-2}+1)m+j = 2^{\ell+1}q+3\cdot 2^{\ell-2}+m+h < 2^{\ell+1}q+3\cdot 2^{\ell-2}+2^{\ell}+2^{\ell-2} < 2^{\ell+1}q+6\cdot 2^{\ell-2},$$

where we have used the first, second, and fifth conditions.

By the above, we have that $xuy = x'vy'$, where $|x| = |x'| = h$ and $|u| = |v|$. Therefore, $u = v$. It follows that the j-fix of $\mathbf{t}$ of length $(r+2^{\ell-2}+1)m$ is not a $(r+2^{\ell-2}+1)$-anti-power, meaning

$$\mathfrak{K}_j(m) \leq r+2^{\ell-2}+1. \qquad \square$$

We are now ready to prove one of the two main results of this section, the proof of which adapts a construction from the proof of [8, Theorem 9].

FIGURE 2. An illustration of the proof of Lemma 4.4 from [8].

Theorem 4.5. *Fix $j \in \mathbb{Z}^{\geq 0}$. For all integers $k \geq 3$, we have $\Gamma_j(k) \leq 3k - 4$. Moreover,* $\limsup_{k\to\infty} \dfrac{\Gamma_j(k)}{k} = 3$.

Proof. Choose an arbitrary integer $k \geq 3$, and let $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(\mathbf{t}, k)$. If $m \leq 5$, then $m \leq 3k - 4$ as desired. We can therefore assume that $m \geq 7$. By Corollary 4.3, we have that $k - 1 \geq 2^{\delta(m)}$, where $\delta(m) = \lceil \log_2(m/3) \rceil$. As m is odd, we have $\delta(m) > \log_2(m/3)$. Therefore, $k - 1 \geq 2^{\delta(m)} > m/3$, meaning $m \leq 3k - 4$. It follows that $\Gamma_j(k) \leq 3k - 4$, which further implies that $\limsup_{k\to\infty}(\Gamma_j(k)/k) \leq 3$.

We now show that $\limsup_{k\to\infty}(\Gamma_j(k)/k) \geq 3$. For each positive integer α, define $k_\alpha = 2^{2\alpha} + 2^\alpha + 2$. Fix an integer $\alpha \geq \lceil \log_2(j) \rceil + 2$, and set $r = 2^\alpha + 1$, $m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$, $\ell = 2\alpha + 2$, $h = j + 1$, $p = 3 \cdot 2^{\alpha-3}$, and $q = 3 \cdot 2^{2\alpha-3} + 2^{\alpha-2}$. It is straightforward to verify that these values of r, m, ℓ, h, p, and q satisfy the first five of the six conditions of Lemma 4.4. Note that the binary expansion of p has exactly two 1's and that the binary expansion of q has exactly three 1's. Therefore, $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$, showing that the sixth and final condition of Lemma 4.4 is also satisfied. We can therefore apply Lemma 4.4 to get that $\mathfrak{K}_j(m) \leq r + 2^{\ell-2} + 1 = k_\alpha$. In other words, we have that the j-fix of $\mathbf{t}$ of length $k_\alpha m$ is not a k_α-anti-power, meaning $\Gamma_j(k_\alpha) \geq m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$. It follows that

$$\frac{\Gamma_j(k_\alpha)}{k_\alpha} \geq \frac{3 \cdot 2^{2\alpha} - 2^\alpha + 1}{2^{2\alpha} + 2^\alpha + 2}$$

for each $\alpha \geq \lceil \log_2(j) \rceil + 2$. Consequently, $(k_\alpha)_{\alpha \geq \lceil \log_2(j) \rceil + 2}$ is an increasing sequence of positive integers with the property that $\Gamma_j(k_\alpha)/k_\alpha \to 3$ as $\alpha \to \infty$. This shows that $\limsup_{k\to\infty}(\Gamma_j(k)/k) \geq 3$, completing the proof.

□

Remark 4.6. The construction in the previous theorem also functions to show that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(k)$ is nonempty for sufficiently large k. In particular, for $j > 0$ and for any integer $\alpha \geq \lceil \log_2(j) \rceil$, we have that $m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1 \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}_j(k)$ for all $k \geq k_\alpha = 2^{2\alpha} + 2^\alpha + 2$.

Next, we present a lemma that will aid in the proof of the final main result of the paper. The lemma adapts constructions from [8, Lemma 10], but it only applies for integers $j > 0$; [8, Lemma 10] gives the same result in the case that $j = 0$.

Lemma 4.7. *Fix $j \in \mathbb{Z}^+$ and let n be the number of 1's in the binary expansion of j. For integers $\alpha \geq \lceil \log_2(j) \rceil + 2$, $\beta \geq \lceil \log_2(j) \rceil + 9$, and $\rho \geq \lceil \log_2(j) \rceil + 8$, define*

$$k_\alpha = 2^{2\alpha} + 2^\alpha + 2 \quad \text{and} \quad K_\beta = 2^{2\beta+1} + 3 \cdot 2^{\beta+3} + 49 \quad \text{and} \quad \kappa_\rho = 2^\rho + 2.$$

We have $\Gamma_j(k_\alpha) \geq 3 \cdot 2^{2\alpha} - 2^\alpha + 1$, $\Gamma_j(K_\beta) \geq 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$, and $\Gamma_j(\kappa_\rho) \geq 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1$, where

$$\chi_j(\rho) = \begin{cases} 2j+1, & \text{if } (n+\rho) \equiv 0 \pmod 2; \\ 4j+3, & \text{if } (n+\rho) \equiv 1 \pmod 2. \end{cases}$$

Proof. The lower bound for $\Gamma_j(k_\alpha)$ was established in the proof of Theorem 4.5. To bound $\Gamma_j(K_\beta)$ from below, let $r = 3 \cdot 2^{\beta+3} + 48$, $m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$, $\ell = 2\beta + 3$, $h = 48 + j$, $p = 9 \cdot 2^\beta + 17$, and $q = 3 \cdot 2^{2\beta-2} + 143 \cdot 2^{\beta-4} + 17$. It is straightforward to verify that these choices of r, m, ℓ, h, p and q satisfy the first five of the six conditions of Lemma 4.4. For the sixth, note that the binary expansion of p has exactly four 1's; using that $\rho \geq 9$, we also see that the binary expansion of q has exactly nine 1's. Therefore, $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$, which shows that the sixth and final condition of Lemma 4.4 is satisfied. Applying Lemma 4.4 gives that $\mathfrak{K}_j(m) \leq r + 2^{\ell-2} + 1 = K_\beta$, meaning the j-fix of $\mathbf{t}$ of length $K_\beta m$ is not a K_β-anti-power. Hence, $\Gamma_j(K_\beta) \geq m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$, as desired.

We now establish the lower bound for $\Gamma_j(\kappa_\rho)$ (recall that $\kappa_\rho = 2^\rho + 2$). Fix $\rho \geq \lceil \log_2(j) \rceil + 8$. Define $r' = 1$, $m' = 5 \cdot 2^{\rho-1} - 8\chi_j(\rho) + 1$, $\ell' = \rho + 2$, $h' = 2^{\rho-1} - 8\chi_j(\rho) + j + 1$, $p' = 0$, and $q' = 5 \cdot 2^{\rho-4} - \chi_j(\rho)$. It is again straightforward to verify that these choices satisfy the first five of the six conditions of Lemma 4.4. To prove that $\mathbf{t}_{p'+1} \neq \mathbf{t}_{q'+1}$, we present an argument that depends on the parity of the number of 1's in the binary expansion of j (which we have denoted by n). Assume that n is odd; the case in which n is even follows similarly. We consider two cases.

First, assume that $\rho \equiv 0 \pmod 2$. In this case, $\chi_j(\rho) = 4j + 3$, so the binary expansion of $\chi_j(\rho)$ has $n + 2$ 1's. Note that

$$\lceil \log_2 \chi_j(\rho) \rceil = \lceil \log_2(4j+3) \rceil \leq 2 + \lceil \log_2(j+1) \rceil \leq 3 + \lceil \log_2(j) \rceil < \rho - 4.$$

It follows that when right-justified, all of the 1's in the binary expansion of $5 \cdot 2^{\rho-4}$ are to the left of all the 1's in the binary expansion of $\chi_j(\rho)$. Binary subtraction therefore shows that there are $\rho - 4 - n$ 1's in the binary expansion of $5 \cdot 2^{\rho-4} - \chi_j(\rho)$. Since n is odd and ρ is even, we get that $\rho - 4 - n$ is odd, meaning $\mathbf{t}_{q'+1} = 1 \neq 0 = \mathbf{t}_{p'+1}$.

Next, assume instead that $\rho \equiv 1 \pmod 2$, meaning $\chi_j(\rho) = 2j + 1$. In this case, the binary expansion of $\chi_j(\rho)$ has $n + 1$ 1's. As before, binary subtraction shows that there are $\rho-3-n$ 1's in the binary expansion of $5\cdot 2^{\rho-4}-\chi_j(\rho)$. Since n is odd and ρ is even, we have that $\rho-3-n$ is odd, meaning $\mathbf{t}_{q'+1}=1\neq 0=\mathbf{t}_{p'+1}$.

We have shown that r', m', ℓ', h', p', and q' satisfy the conditions of Lemma 4.4. Applying the lemma gives that $\mathfrak{K}_j(m)\leq r'+2^{\ell'-2}+1=\kappa_\rho$. Therefore, $\Gamma_j(\kappa_\rho)\geq m=5\cdot 2^{\rho-1}-8\chi_j(\rho)+1$. This completes the proof. □

Theorem 4.8. *For any nonnegative integer j, we have* $\liminf_{k\to\infty}\dfrac{\Gamma_j(k)}{k}=\dfrac{3}{2}$.

Proof. Choose an arbitrary positive integer $k\geq 3$, and let $m=\Gamma_j(k)$. As before, let $\delta(m)=\lceil\log_2(m/3)\rceil$. By Corollary 4.3, we have $k-1\geq 2^{\delta(m)}$. Suppose that k is a power of 2; let us write $k=2^\lambda$. The inequality $k-1\geq 2^{\delta(m)}$ gives that $\delta(m)\leq\lambda-1$. Therefore, $m\leq 3\cdot 2^{\lambda-1}=\dfrac{3k}{2}$. It follows that $\dfrac{\Gamma_j(k)}{k}\leq\dfrac{3}{2}$ whenever k is a power of 2, so $\liminf_{k\to\infty}(\Gamma_j(k)/k)\leq 3/2$.

We now show that $\liminf_{k\to\infty}(\Gamma_j(k)/k)\geq 3/2$. Recall the definitions of k_α, K_β, κ_ρ, and $\chi_j(\rho)$ from Lemma 4.7. Let $\eta=2\lceil\log_2(j)\rceil+21$, fix $k\geq\kappa_\eta$, and put $m=\Gamma_j(k)$. Since $k\geq\kappa_\eta$, Lemma 4.7 and the fact that Γ_j is nondecreasing (see Remark 1.4) together give $m=\Gamma_j(k)\geq\Gamma_j(\kappa_\eta)\geq 5\cdot 2^{\eta-1}-8\chi_j(\eta)+1$. Put $\ell=\lceil\log_2(m+j)\rceil$. Let us first assume that $3\cdot 2^{\ell-2}-2^{(\ell-2)/2}<m+j\leq 2^\ell$. Note that

$$\ell\geq\left\lceil\log_2(5\cdot 2^{\eta-1}-8\chi_j(\eta)+1)\right\rceil\geq\left\lceil\log_2(2^{\eta+1})\right\rceil=\eta+1=2\lceil\log_2 j\rceil+21. \tag{17}$$

In particular, we have that $\ell-1\geq\lceil\log_2 j\rceil+8$. We can therefore apply Lemma 4.7 to get that $\Gamma_j(\kappa_{\ell-1})\geq 5\cdot 2^{\ell-2}-8\chi_j(\ell-1)+1$. Observe that

$$\begin{aligned}5\cdot 2^{\ell-2}-8\chi_j(\ell-1)+1&\geq(m+j)+2^{\ell-2}-8(4j+3)+1\\&\geq(m+j)+\frac{1}{4}\left(5\cdot 2^{\eta-1}-8\chi_j(\eta)+1+j\right)-32j-23\\&\geq(m+j)+\frac{1}{4}\left(5\cdot 2^{2\lceil\log_2 j\rceil+21}-8(4j+3)+j+1\right)-32j-23\\&>m.\end{aligned}$$

It follows that $\Gamma_j(\kappa_{\ell-1})>m$. Because Γ_j is nondecreasing, $\kappa_{\ell-1}>k$. Therefore,

$$\frac{\Gamma_j(k)}{k}>\frac{3\cdot 2^{\ell-2}-2^{(\ell-2)/2}}{\kappa_{\ell-1}}=\frac{3\cdot 2^{\ell-2}-2^{(\ell-2)/2}}{2^{\ell-1}+2} \tag{18}$$

in the case where $3\cdot 2^{\ell-2}-2^{(\ell-2)/2}<m+j\leq 2^\ell$.

Assume next that $2^\ell\leq m+j\leq 3\cdot 2^{\ell-2}-2^{(\ell-2)/2}$ and ℓ is even. By (17), we have $\ell-2>2\lceil\log_2 j\rceil+18$, so

$$(\ell-2)/2>\lceil\log_2 j\rceil+9>\lceil\log_2 j\rceil+2.$$

We can therefore apply Lemma 4.7 to get that $\Gamma_j(k_{(\ell-2)/2}) \geq 3\cdot 2^{\ell-2} - 2^{(\ell-2)/2} + 1 > m$. Because Γ_j is nondecreasing, $k < k_{(\ell-2)/2}$. Thus,

$$\frac{\Gamma_j(k)}{k} > \frac{2^{\ell-1}}{k_{(\ell-2)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 2^{(\ell-2)/2} + 2} \tag{19}$$

in this case.

Finally, assume that $2^{\ell-2} \leq m + j \leq 3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}$ and ℓ is odd. By (17), we have $\ell - 3 \geq 2\lceil \log_2 j \rceil + 18$, so

$$(\ell-3)/2 \geq \lceil \log_2 j \rceil + 9.$$

Lemma 4.7 therefore gives that $\Gamma_j(K_{(\ell-3)/2}) \geq 3 \cdot 2^{\ell-2} - 2^{(\ell-5)/2} + 1 > m$. Since Γ_j is nondecreasing, we have $k < K_{(\ell-3)/2}$. Consequently,

$$\frac{\Gamma_j(k)}{k} > \frac{2^{\ell-1}}{K_{(\ell-3)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 3 \cdot 2^{(\ell+3)/2} + 49} \tag{20}$$

in this case.

By (18), (19), and (20), we have that in all cases,

$$\frac{\Gamma_j(k)}{k} > \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1} + 2}.$$

This gives that $\Gamma_j(k)/k$ is bounded below by a positive function of ℓ. It follows that $\ell \to \infty$ as $k \to \infty$. Consequently, $\liminf_{k\to\infty} \frac{\Gamma_j(k)}{k} \geq \lim_{\ell\to\infty} \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1} + 2} = \frac{3}{2}$. □

5. Conclusion and Further Directions

In Section 4, we proved the exact asymptotic values $\liminf_{k\to\infty}(\Gamma_j(k)/k) = 3/2$ and $\limsup_{k\to\infty}(\Gamma_j(k)/k) = 3$. To better motivate these results and establish a characterization of what could be considered the "interesting" elements of $AP_j(\mathbf{t}, k)$, we would like to have a proof of the conjecture stated in Section 4:

Conjecture 4.2. *For any fixed $j, k \in \mathbb{Z}^{\geq 0}$ with $k \geq 3$, the statement*

$$m \in AP_j(\mathbf{t}, k) \iff 2m \in AP_j(\mathbf{t}, k)$$

holds for all but finitely many $m \in \mathbb{Z}^+$.

We were able to prove exact asymptotic results in Section 4, while in Section 3, we were only able to obtain the asymptotic bounds $\frac{1}{10} \leq \liminf_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{9}{10}$ and $\frac{1}{5} \leq \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} \leq \frac{3}{2}$. However, as of yet, we have no reason to believe that the asymptotic behavior of γ_j and Γ_j depend on j. As such, we extend a conjecture of Defant [8, Conjecture 22] regarding the exact asymptotic growth of γ_0:

Conjecture 5.1. *For any nonnegative integer j, we have*

$$\liminf_{k\to\infty} \frac{\gamma_j(k)}{k} = \frac{9}{10} \quad \text{and} \quad \limsup_{k\to\infty} \frac{\gamma_j(k)}{k} = \frac{3}{2}.$$

Note that Narayanan [12] has proven $\limsup_{k\to\infty}(\gamma_0(k)/k) = 3/2$.

Finally, note that it may be interesting to investigate the properties of $AP_j(x,k)$ for other infinite words x; Defant [8] suggests doing this for $j = 0$. In this paper, we have utilized the recursive structure of $\mathbf{t}$ to prove exact asymptotic values (resp. asymptotic bounds) for $\Gamma_j(k)/k$ (resp. $\gamma_j(k)/k$) that are independent of j. It may be particularly interesting to know whether there are recursively defined infinite words for which the asymptotic growth of analogously defined functions depends on j.

6. Acknowledgements

This work was supported by NSF grant 1650947 and NSA grant H98230-18-1-0010 as a part of the 2018 Duluth Research Experience for Undergraduates (REU). The author would like to thank Niven Achenjang for suggesting the term *j-fix*. The author would also like to thank Prof. Joe Gallian for organizing a wonderful REU, as well as Levent Alpoge, Aaron Berger, and Colin Defant for being outstanding advisors. Finally, the author would like to thank Gallian, Defant, and Andrew Kwon for reading through this manuscript and providing helpful feedback.

References

[1] J.-P. Allouche and H. Cohen. Dirichlet series and curious infinite products, *Bull. Lond. Math. Soc.* **17** (1985) 6, 531-538.

[2] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet-Thue-Morse sequence. *Sequences and their Applications: Proceedings of SETA '98*, (1999), 1-16.

[3] G. Badkobeh, G. Fici, and S.J. Puglisi. Algorithms for anti-powers in strings. *ArXiv e-prints* (May 2018). arXiv: `1805.10042 [cs.DS]`.

[4] S. Brlek. Enumeration of factors in the Thue-Morse word, *Discrete Appl. Math.* **24** (1989) 1, 83-96.

[5] A. Burcroff. (k, λ)-anti-powers and other patterns in words. *ArXiv e-prints* (July 2018). arXiv: `1807.07945 [math.CO]`.

[6] Y. Bugeaud and G.-N. Han. A combinatorial proof of the non-vanishing of Hankel determinants of the Thue-Morse sequence, *Electron. J. Combin.* **21** (2014) 3.

[7] J. Cooper and A. Dutle. Greedy Galois games, *Amer. Math. Monthly* **120** (2013) 5, 441-451.

[8] C. Defant. Anti-power prefixes of the Thue-Morse word, *Electron. J. Combin.* **24** (2017) 1.

[9] F. Dejean. Sur un théorème de Thue, *J. Combin. Theory Ser. A* **13** (1972) 1, 90-99.

[10] G. Fici, A. Restivo, M. Silva, and L. Zamboni. Anti-powers in infinite words, *J. Combin. Theory Ser. A* **157** (2018), 109-119.

[11] K. Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Ann.* **101** (1929), 342-366.

[12] S. Narayanan. Functions on antipower prefix lengths of the Thue-Morse word, *ArXiv e-prints* (May 2017). arXiv: `1705.06310 [math.CO]`.

[13] I. Palacios-Huerta. Tournaments, fairness and the Prouhet-Thue-Morse sequence, *Econ. Inq.* **50** (2012) 3, 848-849.

[14] A. Thue. Über die gegenseitige Lage gleic her Teile gewisser Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912), 1-67.